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Quantitative bounds for the recursive sequence $y_{n+1} = A + \frac{y_n}{y_{n-k}}$

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Abstract

This note provides new quantitative bounds for the recursive equation

$$y_{n+1} = A + \frac{y_n}{y_{n-k}}, \quad n = 0, 1, \ldots,$$

where $y_{-k}, y_{-k+1}, \ldots, y_{-1}, y_0, A \in (0, \infty)$ and $k \in \{2, 3, 4, \ldots\}$. Issues regarding exponential convergence of solutions are also considered. In particular, it is shown that exponential convergence holds for all (A, k) for which global asymptotic stability was proven in [R.M. Abu-Saris, R. DeVault, Global stability of $y_{n+1} = A + \frac{y_n}{y_{n-k}}$, Appl. Math. Lett. 16 (2) (2003) 173–178].

Keywords: Explicit bounds; Difference equation; Stability; Exponential convergence

1. Introduction

Our aim in this note is to examine quantitative behavior of solutions to the equation

$$y_{n+1} = A + \frac{y_n}{y_{n-k}}, \quad n = 0, 1, \ldots, \tag{1}$$

where $y_{-k}, y_{-k+1}, \ldots, y_{-1}, y_0, A \in (0, \infty)$ and $k \in \{2, 3, 4, \ldots\}$.

The study of properties of rational difference equations has been an area of intense interest in recent years; cf. [1,2] and the references therein. Very often the results have stemmed from careful analysis of sign changes and deal with qualitative behavior such as asymptotic stability or periodicity. In real-world applications it may be preferable to have concrete structural information for "small" (non-infinite) n. For some results dealing with the boundedness and persistence of solutions to such equations, cf. [1,3–7], and [8].

In [9], the authors proved some conditions for global asymptotic stability of the positive equilibrium of (1). Here we obtain explicit bounds of the form

$$R_i \le y_i \le S_i, \quad i \ge k+1 \tag{2}$$

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where $\{R_i\}$ and $\{S_i\}$ are independent of the initial values $y_{-k}, y_{-k+1}, \ldots, y_{-1}, y_0$. We also provide conditions for exponential convergence of solutions. As an example we show that when $A = k = 2$, we have

$$2 + \prod_{i=n-3}^{n-2} \left(1 - \left(\frac{12}{17}\right)^{\left[\frac{n+2}{10}\right]}\right) \leq y_n \leq 2 + \prod_{i=n-3}^{n-2} \left(1 + \left(\frac{2}{3}\right)^{2\left[\frac{n-3}{10}\right]+1}\right), \tag{3}$$

for $n \geq 6$, where $[\ \cdot\ ]$ indicates the greatest integer function (see Example 1, below).

The work proceeds as follows. In Section 2, we obtain computable explicit bounds of the form in (2) for all solutions of (1) for fixed (A, k). In many instances the upper and lower bounds converge to the unique equilibrium. In Section 3, exponential convergence of solutions to (1) is examined. As a corollary, it is shown that for all (A, k), for which global asymptotic stability was proven in [9], exponential convergence holds for all solutions.

2. Quantitative bounds for solutions to equation (1)

In this section, we obtain computable explicit bounds for all solutions to (1) which are independent of the initial values $y_{-k}, y_{-k+1}, \ldots, y_{-1}, y_0$.

Suppose $\{y_i\}$ satisfies (1) for some fixed $k \in \{2, 3, \ldots\}$, and as in [9] and [10] consider

$$\gamma_n \stackrel{def}{=} \frac{y_{n+1}}{y_n}, \tag{4}$$

for $n \geq -k$. Note that for $n \geq 0$, we have

$$\gamma_n = \frac{A}{y_n} + \frac{1}{y_{n-k}}. \tag{5}$$

From (1) and (4), we have

$$y_n = A + \gamma_{n-2}\gamma_{n-3}\cdots\gamma_{n-k-1}, \tag{6}$$

for $n \geq 1$, and hence by (5) and (6), $\{\gamma_i\}$ satisfies

$$\gamma_i = \frac{A}{A + \gamma_{i-2}\gamma_{i-3}\cdots\gamma_{i-k-1}} + \frac{1}{A + \gamma_{i-k-2}\cdots\gamma_{i-2k-1}}, \tag{7}$$

for $i \geq k + 1$. Now, suppose that

$$L_i \leq \gamma_i \leq U_i \tag{8}$$

for $-k \leq i \leq N - 1$. Then, from (7), we have

$$L_N \leq \gamma_N \leq U_N \tag{9}$$

where

$$\begin{aligned} L_N &= \frac{A}{A + U_{N-2}U_{N-3}\cdots U_{N-k-1}} + \frac{1}{A + U_{N-k-2}\cdots U_{N-2k-1}} \\ U_N &= \frac{A}{A + L_{N-2}L_{N-3}\cdots L_{N-k-1}} + \frac{1}{A + L_{N-k-2}\cdots L_{N-2k-1}}. \end{aligned} \tag{10}$$

Note that $y_i > A$ for $i \geq 1$, and hence from (7),

$$0 < \gamma_i < 1 + 1/A \tag{11}$$

for $k + 1 \leq i \leq 3k + 1$.

Thus, the problem of bounding (1) is reduced to consideration of the system in (10) with initial values $L_i = 0$, $U_i = 1 + 1/A$, for $k + 1 \leq i \leq 3k + 1$.

The following lemma will be useful.

Lemma 1. *The sequences $\{U_i\}$ and $\{L_i\}$ are nonincreasing and nondecreasing respectively.*

Proof. By assumption, $U_i = U_{i-1} = 1 + 1/A$ and $L_i = L_{i-1} = 0$ for $k+2 \le i \le 3k+1$. Hence suppose that $U_i \le U_{i-1}$ and $L_i \ge L_{i-1}$ for $1 \le i < N$ for some $N \ge 3k+2$. By the induction hypothesis, $U_{N-k-2} \ge U_{N-2}$ and $U_{N-2k-2} \ge U_{N-k-2}$ and thus (10) gives

$$\begin{aligned} L_N &= \frac{A}{A + U_{N-2}U_{N-3}\cdots U_{N-k-1}} + \frac{1}{A + U_{N-k-2}\cdots U_{N-2k-1}} \\ &\ge \frac{A}{A + U_{N-3}\cdots U_{N-k-1}U_{N-k-2}} + \frac{1}{A + U_{N-k-3}\cdots U_{N-2k-1}U_{N-2k-2}} \\ &= L_{N-1}. \end{aligned} \tag{12}$$

A similar argument gives $U_N \le U_{N-1}$, and the lemma follows by induction. □

Now, define the sequence $\{x(i)\}$ via $x(0) = 0$ and

$$x(n) = \frac{A+1}{A + x(n-1)^k}, \tag{13}$$

for $n \ge 1$.

Lemma 1 then gives the following simpler bounds for $\{\gamma_i\}$.

Theorem 1. *We have*

$$L_n^* \le \gamma_n \le U_n^*, \tag{14}$$

for $n \ge k+1$, where $L_n^* = x(2[\frac{n+k}{4k+2}])$ and $U_n^* = x(2[\frac{n-k-1}{4k+2}]+1)$.

Proof. By (8), it suffices to prove that $U_i^* \ge U_i$ and $L_i^* \le L_i$ for $i \ge k+1$. Now, note that $U_i^* = U_i$ and $L_i^* = L_i$, for $k+1 \le i \le 3k+1$. Hence, suppose that $U_i^* \ge U_i$ and $L_i^* \le L_i$, for $k+1 \le i < N$, for some $N > 3k+1$. Then,

$$\begin{aligned} U_N &= \frac{A}{A + L_{N-2}L_{N-3}\cdots L_{N-k-1}} + \frac{1}{A + L_{N-k-2}\cdots L_{N-2k-1}} \\ &\le \frac{A}{A + L_{N-k-1}^k} + \frac{1}{A + L_{N-2k-1}^k} \le \frac{A+1}{A + L_{N-2k-1}^k} \le \frac{A+1}{A + L_{N-2k-1}^{*\,k}} \\ &= \frac{A+1}{A + x\left(2\left[\frac{N-k-1}{4k+2}\right]\right)^k} = x\left(2\left[\frac{N-k-1}{4k+2}\right]+1\right) = U_N^*. \end{aligned} \tag{15}$$

Similar computations lead to the inequality $L_N \ge L_N^*$, and the theorem follows by induction. □

Now, note that

$$x(i)^k = \frac{(A+1)^k}{(A + x(i-1)^k)^k}. \tag{16}$$

Considering the transformation $\beta_i = 1 + \frac{x(i)^k}{A}$, we have $\beta_0 = 1$ and

$$\beta_i = 1 + \frac{(A+1)^k}{A(A + x(i-1)^k)^k} = 1 + \frac{\rho}{\beta_{i-1}^k}, \tag{17}$$

for $i > 0$, where $\rho = \frac{(A+1)^k}{A^{k+1}}$.

The recursive sequence in (17) was studied in [3]. There it was shown that the positive equilibrium of (17) is globally asymptotically stable if and only if

$$\rho \le \frac{k^k}{(k-1)^{k+1}}, \tag{18}$$

i.e.,

$$\frac{(A+1)^k}{A^{k+1}} \le \frac{k^k}{(k-1)^{k+1}}. \tag{19}$$

Since the function $f(x) = (x+1)^k/x^{k+1}$ is decreasing on the interval $(0, \infty)$, we have that (19) is equivalent to $A \geq k-1$. Also, the subsequences $\{\beta_{2i}\}$ and $\{\beta_{2i+1}\}$ of the solution $\{\beta_i\}$ in (17) converge monotonically to the equilibrium. From the definition of β_i, $\{x(2i)\}$ and $\{x(2i+1)\}$ converge monotonically as well, and hence so do the bounds $\{L_i^*\}$ and $\{U_i^*\}$.

The sequences $\{L_i^*\}$ and $\{U_i^*\}$ actually converge exponentially whenever $A > k-1$ as is seen from the following result.

Theorem 2. *If $A > k-1$, then the sequence $\{x(i)\}$ converges exponentially to* 1.

Proof. First, define $\{w(i)\}$ via $x(i) = 1 + w(i)$ for $i \geq 0$, and note that by the above remarks, $\lim_{i\to\infty} w(i) = 0$. Now, for sufficiently large i,

$$\begin{aligned} w(i) &= \frac{A+1}{A+(1+w(i-1))^k} - 1 = \frac{A+1-(A+(1+w(i-1))^k)}{A+(1+w(i-1))^k} \\ &= \frac{-kw(i-1)+O(\mid w(i-1)\mid^2)}{A+1+O(\mid w(i-1)\mid)}. \end{aligned}$$

Hence,

$$\left|\frac{w(i)}{w(i-1)}\right| = \frac{k+O(|w(i-1)|)}{A+1+O(|w(i-1)|)}. \tag{20}$$

The result then follows upon taking the limit as i tends to infinity in (20), and using the assumption $A > k-1$. $\square$

In Theorem 1 of [9] the *constant* bound

$$A < y_n < A + \left(1+\frac{1}{A}\right)^k \tag{21}$$

was obtained for $n \geq 2k+2$. Next, we demonstrate, via an example, how one might obtain quantitative bounds on $\{x(i)\}$ (and hence on $\{y_i\}$ via Theorem 1, (14) and (6)).

Example 1. ($A = k = 2$.) Suppose A and k, are fixed, and set $x(i) = w(i) + 1$ for $i \geq 0$. Employing (13), we have

$$x(n) = \frac{A+1}{A+x(n-1)^k} = \frac{A+1}{A+\left(\frac{A+1}{A+x(n-2)^k}\right)^k}, \tag{22}$$

for $n \geq 2$, and hence,

$$w(n) = \frac{A+1}{A+\left(\frac{A+1}{A+(w(n-2)+1)^k}\right)^k} - 1 = f(w(n-2)) \tag{23}$$

where the function f is defined via

$$f(x) = \frac{A+1}{A+\left(\frac{A+1}{A+(x+1)^k}\right)^k} - 1 \tag{24}$$

for $x > -1$.

Note that f as defined in (24) is increasing for $x > -1$, and $f(0) = 0$. Now, suppose $A = k = 2$. Then,

$$f(x) = \frac{x(12+10x+4x^2+x^3)}{27+24x+20x^2+8x^3+2x^4} \tag{25}$$

and for g defined by $g(x) = f(x)/x$, we have

$$g'(x) = -\frac{18+264x+311x^2+208x^3+72x^4+16x^5+2x^6}{(27+24x+20x^2+8x^3+2x^4)^2} < 0 \tag{26}$$

for $x \geq 0$. Thus $g(x) \leq g(0) = 12/27$ for $x \geq 0$, and

$$\frac{f(c^n)}{c^{n+2}} = \frac{f(c^n)}{c^n}\frac{1}{c^2} \leq \frac{12}{27}\frac{1}{c^2}. \tag{27}$$

Hence, $f(c^n) \leq c^{n+2}$ whenever $c^2 \geq 12/27$ or $c \geq \sqrt{12/27} = 2/3$. Also, for $-1 \leq x \leq 0$, $f(x) < 0$ and both $u(x) = 12 + 10x + 4x^2 + x^3$ and $v(x) = 27 + 24x + 20x^2 + 8x^3 + 2x^4$ are nondecreasing. Thus, by (25),

$$\left|\frac{f(-c^n)}{-c^{n+2}}\right| \leq \left|\frac{u(0)}{v(-1)}\right| \frac{1}{c^2} = \frac{12}{17}\frac{1}{c^2}, \tag{28}$$

Hence, $f(-c^n) \geq -c^{n+2}$ whenever $c^2 \geq 12/17$ or $c \geq \sqrt{12/17}$.

Now, set $h^+(n) = (2/3)^n$ for $n \geq 0$, and $h^-(n) = (12/17)^{n/2}$. Note that $w(0) = -1 = -h^-(0)$ and $w(1) = 1/2 \leq h^+(1)$. Thus, suppose $-1 \leq -h^-(2i) \leq w(2i) \leq 0$ and $0 \leq w(2i+1) \leq h^+(2i+1)$, for $0 \leq i \leq N$, for some $N \geq 0$. Then, we have

$$w(2N+2) = f(w(2N)) \geq f(h^-(2N)) \geq -h^-(2N+2), \tag{29}$$

where the first inequality in (29) follows by induction and the nondecreasing nature of f and the second follows by the preceding discussion. Similarly, we have

$$w(2N+3) = f(w(2N+1)) \leq f(h^+(2N+1)) \leq h^+(2N+3), \tag{30}$$

Thus, by induction (and the fact that $w(i) \geq 0$ for i odd and $w(i) \leq 0$ for i even), we have

$$-\left(\frac{12}{17}\right)^{i/2} \leq w(i) \leq \left(\frac{2}{3}\right)^i, \tag{31}$$

for $i \geq 0$.

Employing Theorem 1 and (31) gives

$$1 - \left(\frac{12}{17}\right)^{\left[\frac{n+2}{10}\right]} \leq \gamma_n \leq 1 + \left(\frac{2}{3}\right)^{2\left[\frac{n-3}{10}\right]+1}, \tag{32}$$

for $n \geq 3$, and finally from (6), we obtain (3) for $n \geq 6$. □

3. Exponential convergence for solutions to equation (1)

In this section we prove the following result on exponential convergence of solutions to (1).

Theorem 3. *Suppose $\{y_i\}$ is a solution to (1), and (A, k) satisfies*

$$(A-1)(A+1)^k + 1 > 0. \tag{33}$$

If $\lim_{i\to\infty} y_i = A + 1$ then the convergence is exponential.

Proof. Suppose $\{y_i\}$ satisfies (1) with $\lim_{i\to\infty} y_i = A + 1$, and set

$$z_i = (1 + A) - y_i \tag{34}$$

for $i \geq 0$. Now, suppose that $\epsilon > 0$ and $N > 2k + 1$ are such that

(i) $C < 1$, where

$$\begin{aligned} C &\stackrel{def}{=} \frac{(A+1+\epsilon)^k + (A+1+\epsilon)^{k-1} + \cdots + (A+1+\epsilon)}{(A+1-\epsilon)^{k+1}} \\ &= \frac{A+1+\epsilon}{A+1-\epsilon}\left(\frac{(A+1+\epsilon)^{k-1} + (A+1+\epsilon)^{k-2} + \cdots + 1}{(A+1-\epsilon)^k}\right) \\ &= \frac{A+1+\epsilon}{A+1-\epsilon}\left(\frac{(A+1+\epsilon)^k - 1}{(A+\epsilon)(A+1-\epsilon)^k}\right). \end{aligned} \tag{35}$$

(ii) $|z_i| \le \epsilon$ for $i \ge N-2k-1$ and
(iii) $|z_i| \le r^{i-(N-2k-1)}$ for $N-2k-1 \le i \le N-1$, where $r = C^{\frac{1}{2k+1}} < 1$.

Note that the existence of $\epsilon > 0$ satisfying (i) is guaranteed by (33).

We then have

$$z_N = (A+1) - y_N = \frac{y_{N-k-1} - y_{N-1}}{y_{N-k-1}} = \frac{z_{N-1} - z_{N-k-1}}{y_{N-k-1}}. \tag{36}$$

Iterating (36), then gives

$$\begin{aligned} z_N &= \frac{\frac{z_{N-2}-z_{N-k-2}}{y_{N-k-2}} - z_{N-k-1}}{y_{N-k-1}} \\ &= \frac{z_{N-2}}{y_{N-k-1}y_{N-k-2}} - \frac{z_{N-k-2}}{y_{N-k-1}y_{N-k-2}} - \frac{z_{N-k-1}}{y_{N-k-1}} \\ &= \frac{\frac{z_{N-3}-z_{N-k-3}}{y_{N-k-3}}}{y_{N-k-1}y_{N-k-2}} - \frac{z_{N-k-2}}{y_{N-k-1}y_{N-k-2}} - \frac{z_{N-k-1}}{y_{N-k-1}} \\ &= \frac{z_{N-3}}{y_{N-k-1}y_{N-k-2}y_{N-k-3}} - \frac{z_{N-k-3}}{y_{N-k-1}y_{N-k-2}y_{N-k-3}} - \frac{z_{N-k-2}}{y_{N-k-1}y_{N-k-2}} - \frac{z_{N-k-1}}{y_{N-k-1}} \\ &\vdots \\ &= \frac{z_{N-k-1}}{\prod_{i=0}^{k} y_{N-k-i-1}} - \sum_{j=0}^{k} \frac{z_{N-k-j-1}}{\prod_{i=0}^{j} y_{N-k-i-1}} \\ &= \frac{z_{N-k-1}\left(1 - \prod_{i=1}^{k} y_{N-k-i-1}\right) + \sum_{j=1}^{k} z_{N-k-j-1} \prod_{i=j+1}^{k} y_{N-k-i}}{\prod_{i=0}^{k} y_{N-k-i}}. \end{aligned} \tag{37}$$

Hence,

$$\begin{aligned} |z_N| &\le \frac{r^k \left|1 - \prod_{i=1}^{k} y_{N-k-i-1}\right| + \sum_{j=1}^{k} r^{k-j} \prod_{i=j+1}^{k} y_{N-k-i-1}}{\prod_{i=0}^{k} y_{N-k-i-1}} \\ &\le \left(\frac{\left|1 - \prod_{i=1}^{k} y_{N-k-i-1}\right| + \sum_{j=1}^{k} \prod_{i=j+1}^{k} y_{N-k-i-1}}{\prod_{i=0}^{k} y_{N-k-i-1}} \right) \\ &\le \frac{(A+1+\epsilon)^k + (A+1+\epsilon)^{k-1} + \cdots + (A+1+\epsilon) + 1 - 1}{(A+1-\epsilon)^{k+1}} \\ &= C = r^{2k+1} = r^{N-(N-2k-1)}. \end{aligned} \tag{38}$$

By induction, we have $|z_i| \le r^{i-(N-2k-1)}$ for all $i > N-2k-1$ and the result follows. □

Note that all (A, k) for which global asymptotic stability was proven in [9], satisfy (33), and in particular we have the following.

Corollary 1. *All solutions to* (1) *converge exponentially to* $A+1$ *whenever any of the following hold.*

(1) $k = 2$ *and* $A > \frac{\sqrt{5}-1}{2}$,
(2) $k = 3$ *and* $A > \frac{q}{3} + \frac{4}{3q} - \frac{2}{3}$, *where* $q = (19 + 3\sqrt{33})^{1/3}$,
(3) $k > 3$ *and* $A > 1$.

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